Internal Direct product

As we have seen, the external direct product provides a method of putting groups together to get a larger group in such a way that we can determine many properties of the larger group from the properties of the component pieces. That means

(i) Let $G = H \oplus K$ then we get $|G| = |H||K|$.

(ii) every element of G is of the form (h,k) where $h \in H, k \in K$.

(iii) if $|h|$ and $|k|$ are finite, then $|(h,k)| = l.c.m(|h|,|k|)$.

(iv) if H and K are abelian then G is abelian.

(v) if H and K are cyclic and $g.c.d(|H|,|K|) = 1$ then $G = H \oplus K$ is cyclic.

It would be quite useful to be able to reverse this process, i.e, to be able to start with a large group G and break it down into a product of subgroups in such a way that we easyly study many properties of G from each component H and K. Now we define internal Direct product.

Defn: (IDP).

Let H and K are two normal subgroup of G and.

(i) $G = HK$

(ii) $H \cap K = \{e\}$

Then G is the internal direct product of H and K. which is denoted by $G = H \times K$.

Note: (i) In IDP we starting with a large group G and we want to find two normal subgroup H and K of G s.t G is external direct product of H and K. where H and K are related or not.

(ii) Difference between IDP and EDP is that the IDP can be fromed within G itself, using groups of G and the operation of G, but the external direct product can be formed with totally unrelated groups by creating a new set and a new operation.

Defn (General Defn of IDP).

Let $\{H_i \mid i \in I_n\}$ be a finite collection of normal subgroup of G. we say that G is IDP of $H_1, H_2 \cdots H_n$ and we write $G = H_1 \times H_2 \times \cdots \times H_n$, if (i) $G = H_1 H_2 \cdots H_n$

(ii) $H_i \cap (H_1 H_2 \cdots H_{i-1} H_{i+1} \cdots H_n) = \{e\} \; \forall i$

Scanned by CamScanner

Example ① Let $G = K_4 = \{e, a, b, ab = ba = c\}$. Let $H_1 = \{e, a\}$
$H_2 = \{e, b\}$. Now $[G : H_1] = \frac{4}{2} = 2 \Rightarrow H_1 \trianglelefteq G$.

$\therefore$ Also $[G : H_2] = \frac{4}{2} = 2 \Rightarrow H_2 \trianglelefteq G$.

Now $H_1 \cap H_2 = \{e\}$. So $G = H_1 \times H_2$ i.e.
G is internal direct product of its normal subgroup H_1 and H_2.

② Let $G = D_6 = \{R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}, F_1, F_2, F_3, F_4, F_5, F_6\}$

$H_1 = \{R_0, R_{120}, R_{240}, F, R_{120}F, R_{240}F\}$

$H_2 = \{R_0, R_{180}\}$.

Then $H_1 \trianglelefteq G$ and $H_2 \trianglelefteq G$.

Also $H_1 \cap H_2 = \{R_0\}$.

$\therefore$ G is IDP of H_1 and H_2 i.e. $G = H_1 \times H_2$.

③ Let $H_1 = \{e, (12)\}$, $H_2 = \{e, (123), (132)\}$ are two subgroup of S_3. Is $S_3 = H_1 \times H_2$

Soln clearly $H_1H_2 = \{e, (123), (132), (12), (12)(123), (12)(132)\}$

$= \{e, (123), (132), (12), (23), (13)\}$

$= S_3$

$S_3 = H_1H_2$ Also $H_1 \cap H_2 = \{e\}$.

But $H_1H_2 \neq H_2H_1$ i.e. H_2 is not normal in G.

because $(12)(132) = (13)$;

$(132)(12) = (23)$

$\therefore (12)(132) \neq (132)(12)$.

Another way

Let $S_3 \equiv H_1 \times H_2$ then $|H_1| = 2$, $|H_2| = 3$

$\therefore |H_1 \times H_2| = 6$ g.c.d$(|H_1|, |H_2|) = 1$

$\Rightarrow H_1 \times H_2$ is cyclic

but S_3 is not cyclic so $S_3 \neq H_1 \times H_2$

Scanned by CamScanner

Th: Let G be a group and H, K are two subgroup of G. If G is a internal direct product of H and K then (i) $G \simeq H \times K$

(ii) $G/H \simeq K$ and $G/K \simeq H$.

(H.T.)

Th: If a group G is the internal direct product of a finite number of subgroups $H_1, H_2, \ldots, H_n$, then G is isomorphic to the external direct product of $H_1, H_2, \ldots, H_n$.

i.e. $H_1 \times H_2 \times H_3 \times \ldots \times H_n \simeq H_1 \oplus H_2 \oplus \cdots \oplus H_n$.

proof: Since $G = H_1 \times H_2 \times \cdots \times H_n$ so $\forall i = 1, 2 \ldots n$, H_i's are normal subgroup of G and $G = H_1 H_2 \cdots H_n$ and

$$H_i \cap \{H_1 H_2 \cdots H_{i-1} H_{i+1} \cdots H_n\} = \{e\}.$$

Then $\forall g \in G$, $\exists\, h_i \in H_i\ \forall i$ s.t $g = h_1 h_2 \cdots h_n$. We show

(i) $\forall h_i \in H_i, \forall h_j \in H_j$, $h_i h_j = h_j h_i$, $i, j = 1, 2 \ldots n$. $(i \neq j)$

(ii) $g = h_1 h_2 \cdots h_n$ this presentation is unique.

For i: if $h_i \in H_i$, $h_j \in H_j$ with $i \neq j$

$(h_i h_j h_i^{-1}) h_j^{-1} \in H_j h_j^{-1} = H_j$

$h_i (h_j h_i^{-1} h_j^{-1}) \in h_i H_i = H_i$

$\Rightarrow h_i h_j h_i^{-1} h_j^{-1} \in H_i \cap H_j = \{e\}$

$\Rightarrow h_i h_j h_i^{-1} h_j^{-1} = e$.

$\Rightarrow h_i h_j = h_j h_i$

For (ii): Let $\exists\, h_i \in H_i\ \forall i = 1, 2 \ldots n$. $h_i' \in H_i\ \forall i = 1, 2 \ldots n$

s.t $g = h_1 h_2 \cdots h_n = h_1' h_2' \cdots h_n'$

$\Rightarrow h_n' h_n^{-1} = (h_1')^{-1} h_1 (h_2')^{-1} h_2 \cdots (h_{n-1}')^{-1} h_{n-1}$

$\Rightarrow h_n' h_n^{-1} \in H_1 H_2 \cdots H_{n-1} \cap H_n = \{e\}$

Scanned by CamScanner

$\Rightarrow h_n' h_n^{-1} = e$.

$\Rightarrow h_n' = h_n$

Similarly $h_i' = h_i \quad \forall i = 1, 2 \cdots n$.

So $g = h_1 h_2 \cdots h_n$, $\forall h_i \in H_i \quad i = 1, 2 \cdots n$.

is unique.

For $H_1 \times H_2 \times \cdots \times H_n \simeq H_1 \oplus H_2 \oplus \cdots \oplus H_n$. we define a function

$\varphi : \underset{(G)}{\underline{H_1 \times H_2 \times \cdots \times H_n}} \longrightarrow H_1 \oplus H_2 \oplus \cdots \oplus H_n$ by

$\varphi(h_1 h_2 \cdots h_n) = (h_1, h_2, \cdots h_n) \quad \forall g = h_1 h_2 \cdots h_n \in G$

<u>Homomorphism</u>:

Let $g_1 = h_1 h_2 \cdots h_n$, $g_2 = h_1' h_2' \cdots h_n'$

Then $\varphi(g_1 g_2) = \varphi(h_1 h_2 \cdots h_n h_1' h_2' \cdots h_n')$.

$= \varphi((h_1 h_1')(h_2 h_2') \cdots (h_n h_n'))$ by (i).

$= (h_1 h_1', h_2 h_2', \cdots, (h_n h_n'))$.

$= (h_1, h_2, \cdots h_n)(h_1', h_2', \cdots h_n')$.

$= \varphi(g_1) \varphi(g_2)$.

$\therefore \varphi$ is homomorphism.

<u>one-one</u>: Let $g_1, g_2 \in G$ s.t $\varphi(g_1) = \varphi(g_2)$

$\Rightarrow (h_1, h_2 \cdots h_n) = (h_1', h_1' \cdots h_1'^n)$

$\Rightarrow h_i = h_i' \quad \forall i = 1, 2 \cdots n$.

$\Rightarrow g_1 = g_2$ [$\because$ presentation of g_1 & g_2 is unique]

<u>onto</u>: Let $(h_1, h_2, \cdots h_n) \in H_1 \oplus H_2 \oplus \cdots \oplus H_n$

then by the defn $\exists (h_1 h_2 \cdots h_n) \in G$ s.t

$\varphi(g) = (h_1, h_2, \cdots h_n)$

$\therefore \varphi$ is isomorphism

$\therefore H_1 \times H_2 \times \cdots \times H_n \simeq H_1 \oplus H_2 \oplus \cdots \oplus H_n$

Scanned by CamScanner

Th: Every group of order p^2 (where p is prime), is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

proof: Let $|G| = p^2$ where p is prime. Let $a \in G$ be a non identity element of G. Then possible order of a are $1, p, p^2$ $[\because o(a) \mid o(G)]$.

if $o(a) = 1$ then $a = e$ but we take $a \neq e$.

$\therefore o(a) \neq 1$.

if $o(a) = p^2$ then G has an element of order $p^2 = |G|$

so G is cyclic. Therefore $G \simeq \mathbb{Z}_{p^2}$

if $o(a) = p$. Let $H = \langle a \rangle$. which is cyclic subgroup of G.

Then $H \simeq \mathbb{Z}_p$.

We show that H is normal in G

Let H is not normal in G. Then $\exists\, b \in G$ s.t

$bab^{-1} \notin H$. $\therefore \langle bab^{-1} \rangle = K$ is a subgroup of G.

$\therefore o(bab^{-1}) = 1, p, p^2$, if $o(bab^{-1}) = 1$

$\Rightarrow bab^{-1} = e$

$\Rightarrow ba = b$

$\Rightarrow a = e$. which is impossible.

$\therefore o(bab^{-1}) = p^2 \Rightarrow G \simeq \mathbb{Z}_{p^2}$

if $o(bab^{-1}) = p$.

Since H and K are subgroup so $H \cap K$ is a subgroup of G, H and K.

$\Rightarrow H \cap K = \{e\}$. $[\because H \cap K$ is subgroup of H and $K]$

$\therefore$ distinct left cosets of H are $H, aH, a^2H, \ldots a^{p-1}H$.

since b^{-1} must lie in one of these cosets so

$b^{-1} = a^i (bab^{-1})^j = a^i b a^j b^{-1}$ for some i, j

$\Rightarrow e = a^i b a^j \Rightarrow b = a^{-i-j} \in H$. which is contradiction

Scanned by CamScanner

$\therefore$ H is normal in G. Let $b \in G$ and $b (\notin \langle a \rangle) \neq e$.

$\therefore$ $o(b) = p$ and $\langle a \rangle \cap \langle b \rangle = \{e\}$.

Then $G = \langle a \rangle \times \langle b \rangle \simeq \langle a \rangle \oplus \langle b \rangle$

$$\simeq \mathbb{Z}/p \oplus \mathbb{Z}/p.$$

Corollary: If G is a group of order p^2, where p is prime then G is abelian. (H T)

prob ① Show that $G/H \not\simeq G/K$ when $G = \mathbb{Z}_4 \oplus U(4)$.

$H = \langle (2,3) \rangle$, $K = \langle (2,1) \rangle$.

② suppose that G is non-abelian group of order p^3 where p is a prime, and $Z(G) \neq e$. prove that $|Z(G)| = p$.

③ $G/Z(G) \simeq Inn(G)$.

Scanned by CamScanner